Vectors

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Note : The notes given in this file is no substitute to the much detailed discussion held in the online/contact classes with active participation of students. It , at best, serves the purpose of ready reference for important concepts/derivations covered in the classes.

Scalar quantities (or scalars)

Quantities that have only magnitude and no direction associated with them are called scalar quantities.

Eg : distance, speed, temperature, density, volume, electric current …

Vector quantities (or vectors)

Quantities that have magnitude, direction and follow vectors laws of addition are called vector quantities.

Eg : displacement, velocity, acceleration, force, momentum …

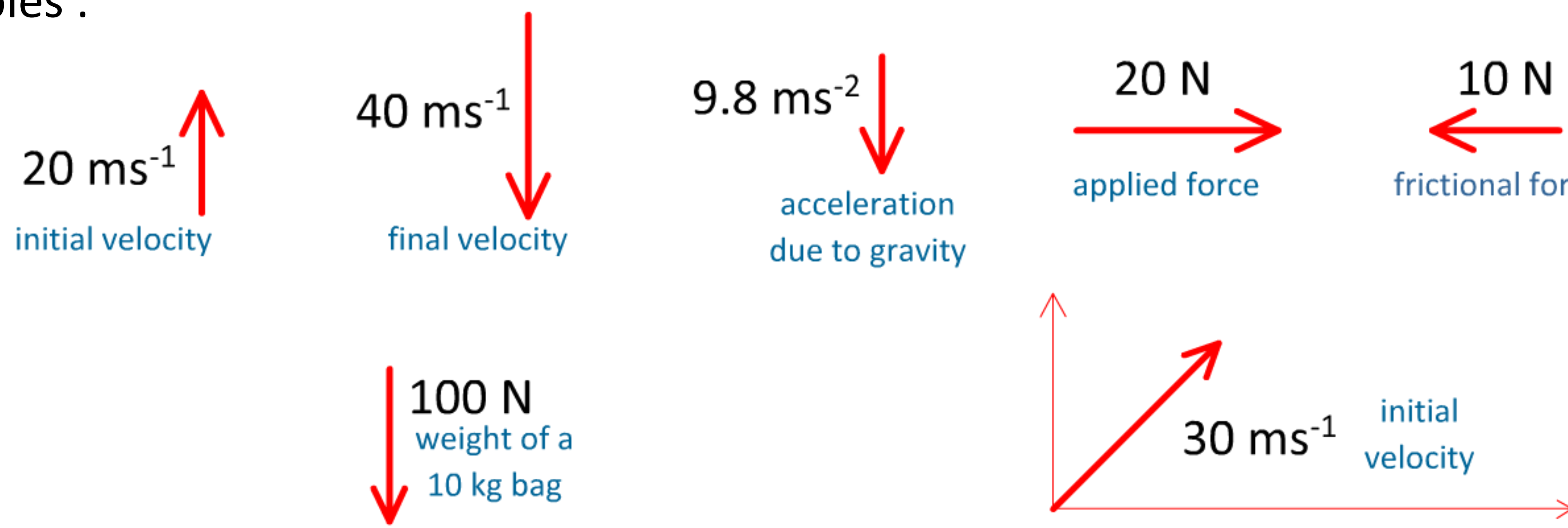
Graphical representation of vectors

Graphically, a vector is represented by a directed line segment (an arrow).

Length of the arrow is an indication of magnitude of the vector.

Direction of arrowhead is in the direction of the vector .

Examples :



Graphical representations are useful only for a quick visualization / estimate of the phenomenon

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Textual representation of vectors

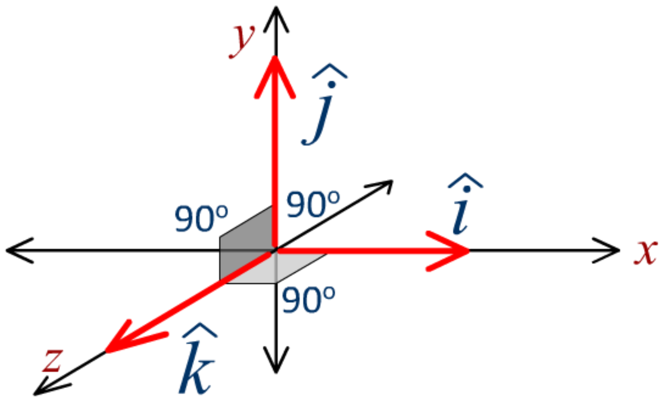
Textually (while writing) a vector is represented with a bar over the symbol representing the vector quantity.

Textually (in print) a vector is represented with a bold face for the symbol representing the vector quantity

Examples :

	In print	Hand written
displacement	$\boldsymbol{S}$	$\overline{S}$
velocity	$\boldsymbol{v}$	$\overline{v}$
acceleration	$\boldsymbol{a}$	$\overline{a}$
force	$\boldsymbol{F}$	$\overline{F}$

Types of vectors



- **Unit vector** : A vector whose magnitude is unity (or 1) called a unit vector.

  It is denoted by a cap on the symbol representing the vector.

  It is used to represent a specific direction.

- **Position vector** : A vector drawn from the origin to the instantaneous position of the body is called a position vector.

  It is generally denoted by ***r***.

- **Negative of a vector** : Negative of a vector is a vector of same magnitude but opposite direction.

- **Null vector** : A vector of zero magnitude and unspecified direction is called a null vector.

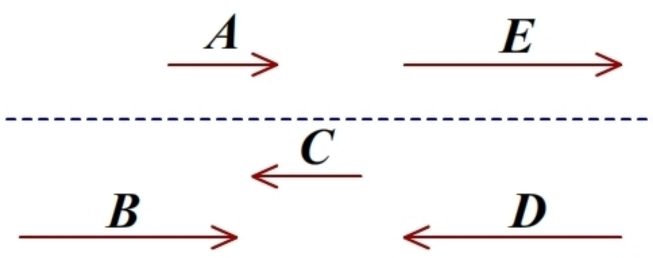
  It is denoted as **O**.

Types of vectors

- Equal vectors : Two vectors are said to be equal if they have same magnitude and direction
- Parallel vectors : Two vectors are said to be parallel to each other if they are parallel to the same reference line
- Like vectors : Two parallel vectors are said to be like if they are along the same direction.
- Unlike vectors : Two parallel vectors are said to be unlike if they are in opposite directions.

Examples

- $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{B}$ are like vectors
- $\boldsymbol{C}$ and $\boldsymbol{D}$ are like vectors
- $\boldsymbol{A}$ and $\boldsymbol{D}$ are unlike vectors
- ALL the vectors are parallel vectors



Types of vectors

- Polar vector : A vector whose direction is reversed when the direction of the coordinate axes are reversed is called a polar vector.

  Eg. Velocity, displacement etc.

- Axial vector : A vector whose direction is not reversed when the coordinate axes are reversed is called an axial vector. Such a vector is also called a pseudo-vector.

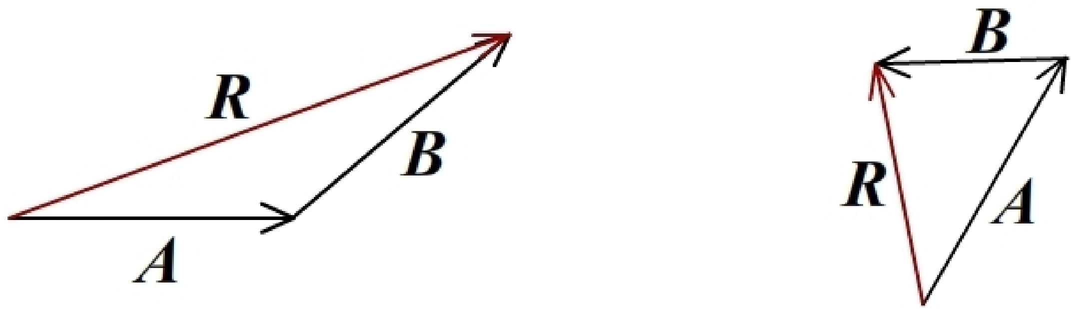
  Eg. Angular momentum etc.

Vector addition (graphical methods)

Triangle law of addition of vectors

If two vectors represent two sides of a triangle taken in an order, both in magnitude and direction, then their resultant is given by the closing side of the triangle, both in magnitude and direction, taken in the reverse order.

Examples :

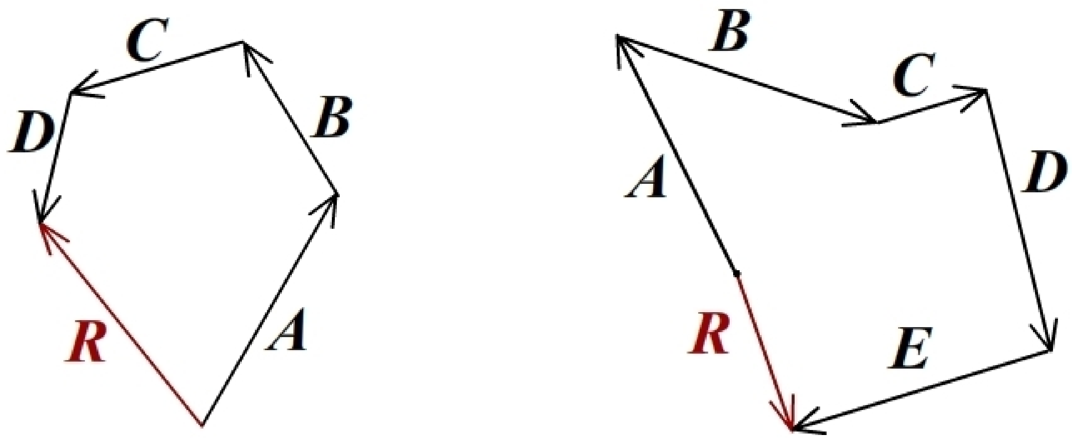


Vector addition (graphical methods)

Polygon law of addition of vectors

If a set of vectors represent adjacent sides of a polygon taken in an order, both in magnitude and direction, then their resultant is given, both in magnitude and direction, by the closing side of the polygon taken in the reverse order.

Examples :



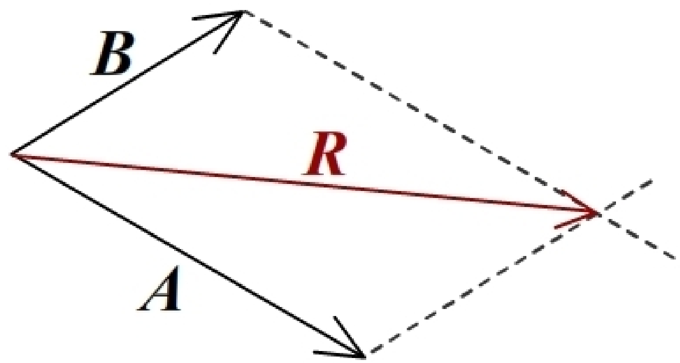
Vector addition (graphical methods)

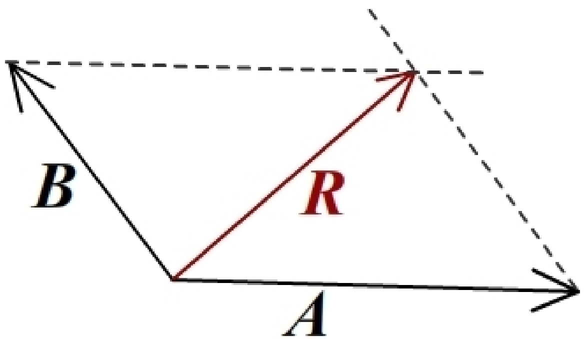
Click here for simulation

Parallelogram law of addition of vectors

If two vectors represent adjacent sides of a parallelogram, both in magnitude and direction, then their resultant is given, both in magnitude and direction, by the diagonal of the parallelogram drawn from the same point.

Examples :





Parallelogram law of addition of vectors (magnitude)

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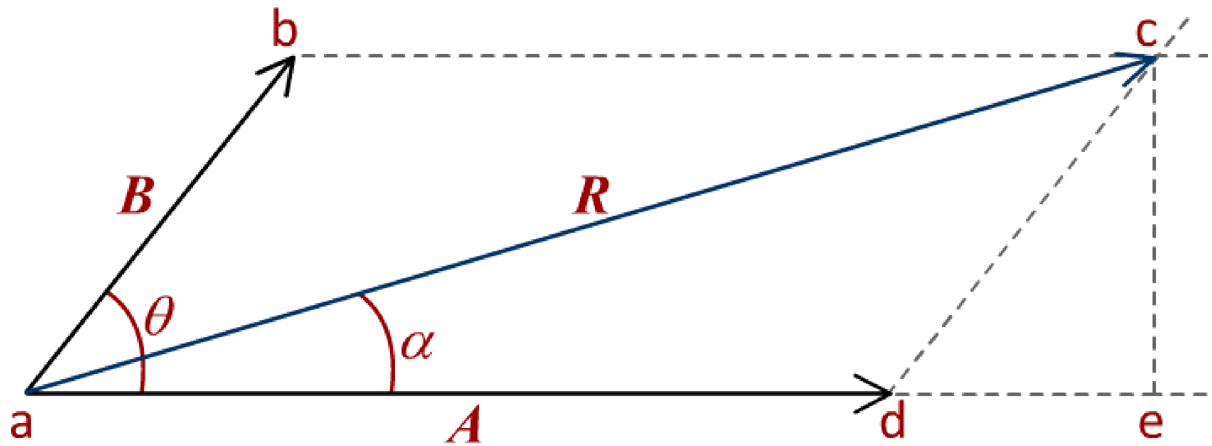
Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two vectors and θ be the angle between them. A line is drawn parallel to $\boldsymbol{A}$ and another line is drawn parallel to $\boldsymbol{B}$. A perpendicular is dropped from the point of intersection.



From Δ ace we get

$$ac^2 = ae^2 + ec^2$$

$$ac^2 = (ad + de)^2 + ec^2$$

$$ac^2 = ad^2 + de^2 + 2\,ad\,de + ec^2 \qquad \text{i}$$

From Δ dce we get

$$\sin(\theta) = \frac{ec}{B} \Rightarrow ec = B\sin(\theta) \qquad \text{ii}$$

$$\cos(\theta) = \frac{de}{B} \Rightarrow de = B\cos(\theta) \qquad \text{iii}$$

Substituting eqs(ii) and (iii) in eq(i)

$$R^2 = A^2 + B^2\cos^2(\theta) + 2AB\cos(\theta) + B^2\sin^2(\theta)$$

$$R^2 = A^2 + B^2 + 2AB\cos(\theta)$$

$$\boxed{R = \sqrt{A^2 + B^2 + 2AB\cos(\theta)}}$$

Parallelogram law of addition of vectors (direction)

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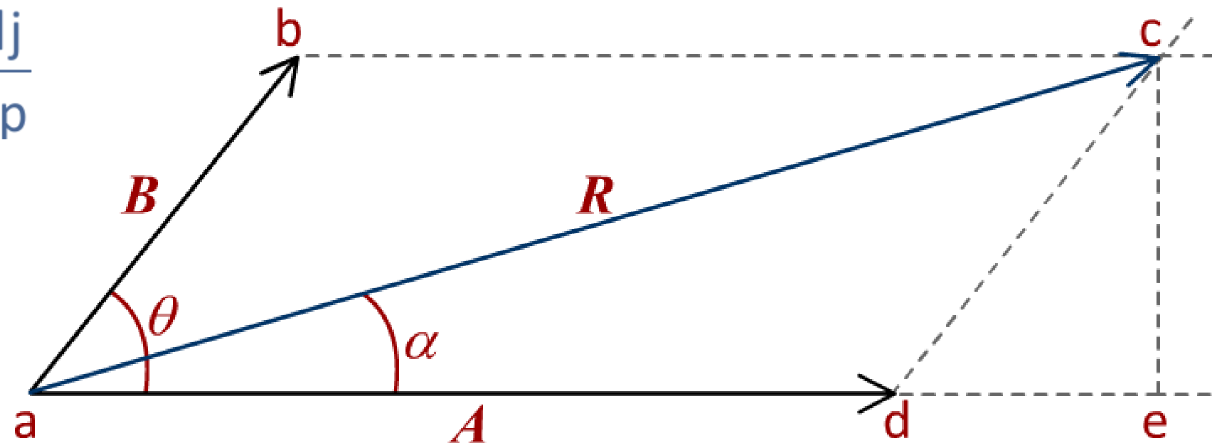
$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \cos(\theta) = \frac{\text{adj}}{\text{hyp}}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$



From Δ ace we get

$$\tan(\alpha) = \frac{ec}{ae}$$

$$\tan(\alpha) = \frac{ec}{(ad + de)} \quad \text{i}$$

$$ec = B\sin(\theta) \quad \text{ii}$$

$$de = B\cos(\theta) \quad \text{iii}$$

Substituting eqs(ii) and (iii) in eq(i)

$$\tan(\alpha) = \frac{B\sin(\theta)}{A + B\cos(\theta)}$$

$$\boxed{\alpha = \tan^{-1}\left[\frac{B\sin(\theta)}{A + B\cos(\theta)}\right]}$$

Subtraction of two vectors

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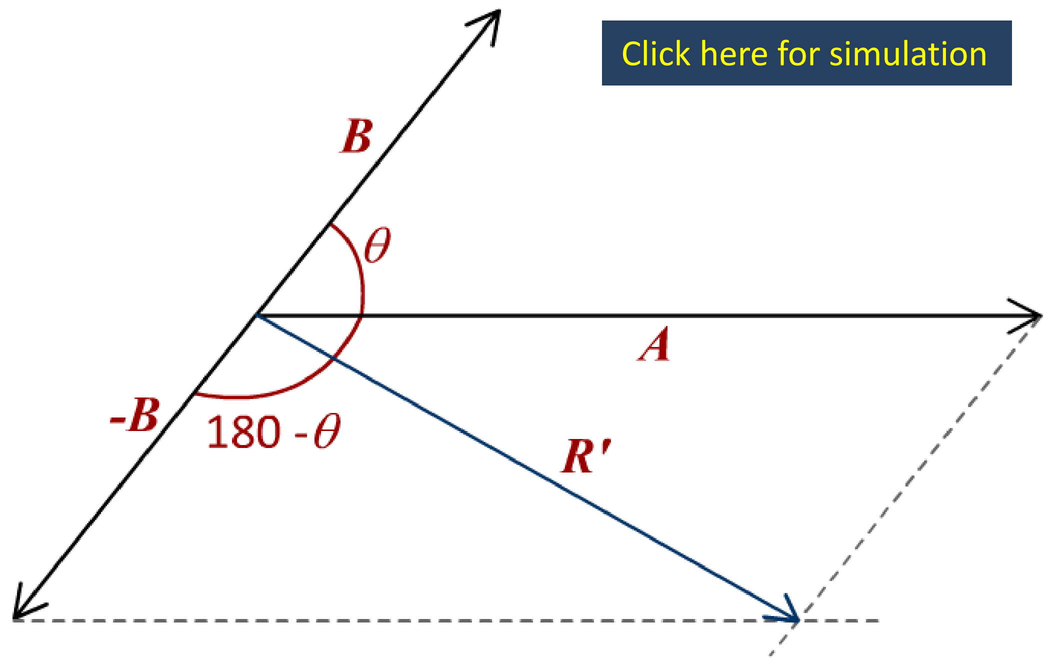
$$\boldsymbol{R'} = \boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$$

Take negative of the second vector and then add the vectors using parallelogram law !



$$R = \sqrt{A^2 + B^2 + 2AB\cos(\theta)}$$

$$R' = \sqrt{A^2 + B^2 + 2AB\cos(180-\theta)}$$

$$R' = \sqrt{A^2 + B^2 - 2AB\cos(\theta)}$$

$$\alpha' = \tan^{-1}\left[\frac{B\sin(180-\theta)}{A + B\cos(180-\theta)}\right]$$

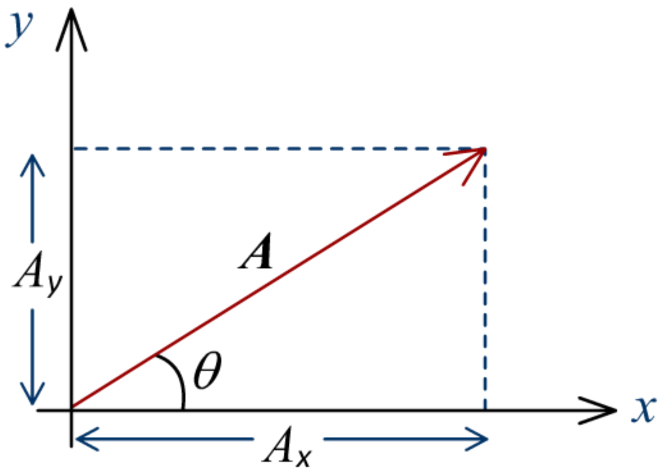
$$\alpha' = \tan^{-1}\left[\frac{B\sin(\theta)}{A - B\cos(\theta)}\right]$$

Resolution of vectors

Resolution of vectors is a technique to obtain the "*effect*" or "*contribution*" of a vector in specific direction(s).

The "*effect*" or "*contribution*" of a vector in a specific direction is called <u>component of the vector</u> in that direction.

Consider a vector $\boldsymbol{A}$ of magnitude $|A|$ making an angle of θ w.r.t. the x-axis as shown in the figure.



A_x : x-component A_y : y-component

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$$\cos(\theta) = \frac{A_x}{|A|}$$

$$\Rightarrow A_x = |A|\cos(\theta) \qquad \text{(i)}$$

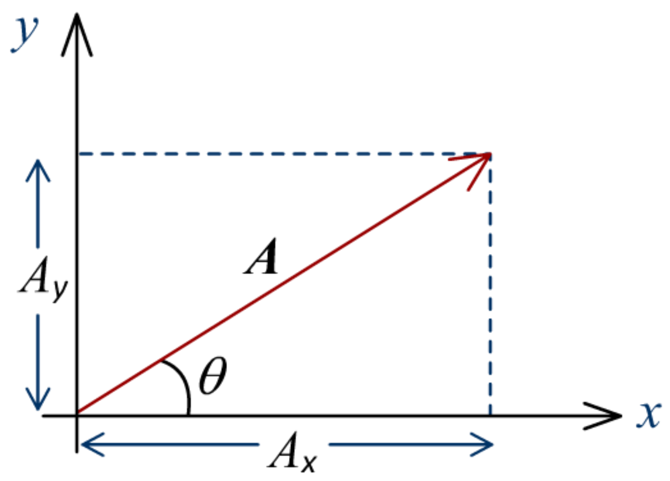
$$\sin(\theta) = \frac{A_y}{|A|}$$

$$\Rightarrow A_y = |A|\sin(\theta) \qquad \text{(ii)}$$

Using equations (i) & (ii) a vector is represented in its component form as

$$\boxed{\boldsymbol{A} = A_x\,\hat{i} + A_y\,\hat{j}}$$

Resolution of vectors



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A vector may be resolved into 2 (or 3) components in any direction. The directions in which a vector is resolved depends on the problem. You will use different vector resolutions in projectile motion, motion on a inclined plane, movement of charged particle in $\boldsymbol{E}$ and $\boldsymbol{B}$ etc.

$$A_x = |A|\cos(\theta) \quad \text{i}$$

$$A_y = |A|\sin(\theta) \quad \text{ii}$$

Equations (i) and (ii) are used to obtain components if magnitude (A) and direction (θ) are known

$$\boxed{|\boldsymbol{A}| = \sqrt{A_x^2 + A_y^2}} \quad \text{iii}$$

$$\boxed{\tan(\theta) = \frac{A_y}{A_x}} \quad \text{iv}$$

Equations (iii) and (iv) are used to obtain magnitude (A) and direction (θ) if components (A_x and A_y) are known

Resolution of vectors

- In three dimensions, a vector is given by $\boldsymbol{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$
- Unit vector in the direction of a given vector is obtained by dividing a vector by its own magnitude.

$$\boxed{\hat{A} = \frac{A_x \hat{i} + A_y \hat{j}}{|A|}}$$

- Component of a vector may be positive, negative or zero
- Magnitude of a vector is always positive
- For a vector of 2 or 3 dimensions, algebraic sum of components of a vector is NOT equal to the magnitude of the vector
- Each component of a *unit vector* is not 1
- A vector may be resolved into components other than x, y and z components

Analytical method of addition of vectors

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two vectors in a plane given in their component form as

$$\boldsymbol{A} = A_x\,\hat{i} + A_y\,\hat{j} \qquad \boldsymbol{B} = B_x\,\hat{i} + B_y\,\hat{j}$$

Resultant of addition of $\boldsymbol{A}$ and $\boldsymbol{B}$ is given by

$$\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B}$$

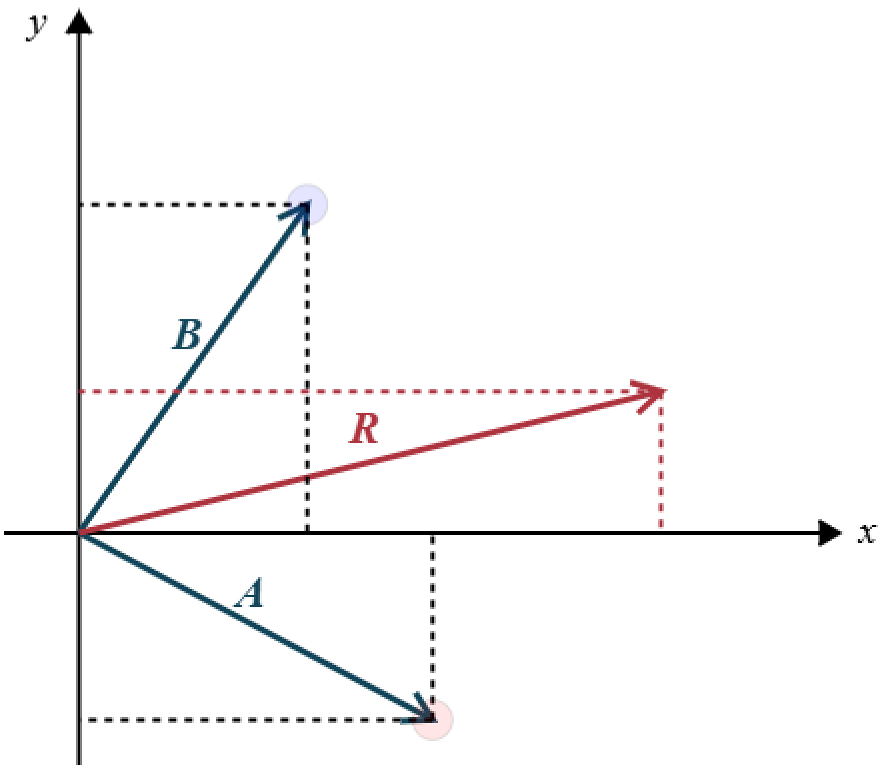
$$\boldsymbol{R} = \left(A_x + B_x\right)\hat{i} + \left(A_y + B_y\right)\hat{j}$$

Magnitude of the resultant is given by

$$|\,\boldsymbol{R}\,| = \sqrt{\left(A_x + B_x\right)^2 + \left(A_y + B_y\right)^2}$$

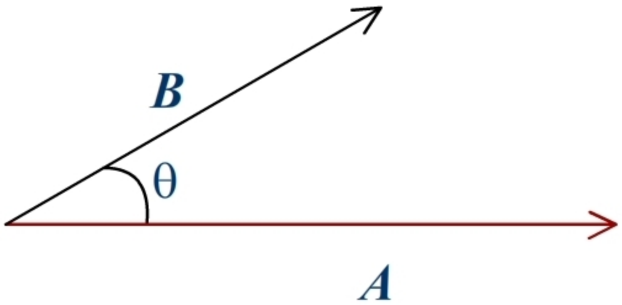
Direction of the resultant is given by

$$\theta = \tan^{-1}\left(\frac{\left(A_y + B_y\right)}{\left(A_x + B_x\right)}\right)$$



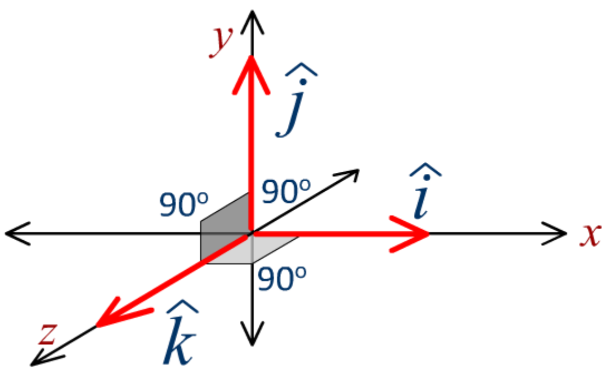
Scalar product or dot product of vectors

Scalar product of two vectors is defined as the product of magnitudes of the two vectors and cosine of angle between the vectors.



$$\boxed{\boldsymbol{A} \bullet \boldsymbol{B} = |A||B|\cos(\theta)}$$

If the vectors are given in their component forms then we get



$$\boldsymbol{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\boldsymbol{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\boxed{\boldsymbol{A} \bullet \boldsymbol{B} = (A_xB_x + A_yB_y + A_zB_z)}$$

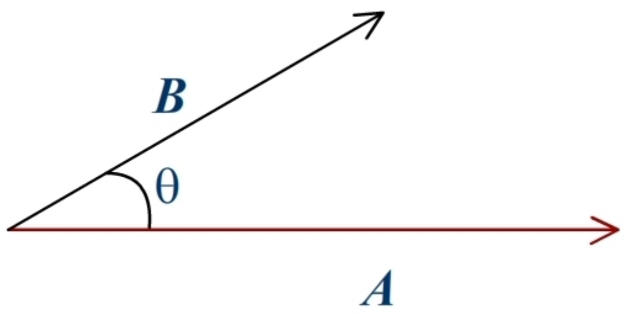
Properties of scalar product

- Dot product of two parallel vectors is equal to the product of the magnitudes of the vectors
- Dot product of two mutually perpendicular vectors is zero
- Dot product is commutative in nature $\boldsymbol{A} \bullet \boldsymbol{B} = \boldsymbol{B} \bullet \boldsymbol{A}$
- Dot product is distributive $\boldsymbol{A} \bullet (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \bullet \boldsymbol{B} + \boldsymbol{A} \bullet \boldsymbol{C}$
- $\hat{i} \bullet \hat{j} = \hat{j} \bullet \hat{k} = \hat{k} \bullet \hat{i} = 0$
- $\hat{i} \bullet \hat{i} = \hat{j} \bullet \hat{j} = \hat{k} \bullet \hat{k} = 1$
- For a real number (a constant) λ we get $\boldsymbol{A} \bullet (\lambda \boldsymbol{B}) = \lambda \boldsymbol{A} \bullet \boldsymbol{B}$

Angle between two vectors

$$\boldsymbol{A} \bullet \boldsymbol{B} = |A||B|\cos(\theta) \quad \text{i}$$

$$\boldsymbol{A} \bullet \boldsymbol{B} = (A_x B_x + A_y B_y + A_z B_z) \quad \text{ii}$$



$$|A||B|\cos(\theta) = (A_x B_x + A_y B_y + A_z B_z)$$

$$\boxed{\cos(\theta) = \frac{(A_x B_x + A_y B_y + A_z B_z)}{|A||B|}}$$

Vector product or cross product of vectors

Magnitude of cross product of two vectors is defined as the product of magnitudes of the two vectors and sine of angle between the vectors.

$$\boldsymbol{A} \times \boldsymbol{B} = |A||B|\sin(\theta)\,\hat{n}$$

Direction is given by right hand thumb rule

$\hat{n}$ is a the unit vector along the resultant, given by RHT rule

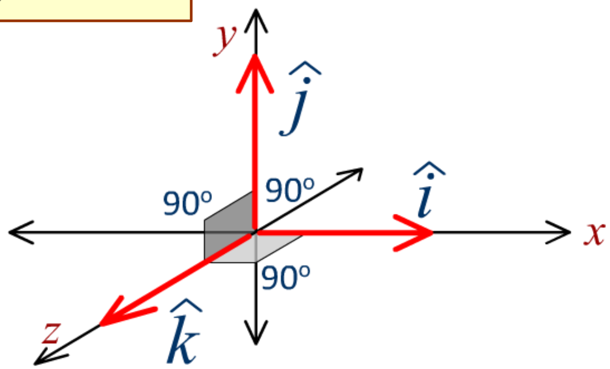
If the vectors are given in their component forms i.e.

$$\boldsymbol{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \boldsymbol{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

then the cross product is given by

$$\boldsymbol{A} \times \boldsymbol{B} = (A_yB_z - A_zB_y)\hat{i} - (A_xB_z - A_zB_x)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$$

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



Properties of vector product (or cross product)

- Cross product of two parallel vectors is a null vector
- Magnitude of cross product of two mutually perpendicular vectors is equal to the product of their magnitudes
- Cross product is not commutative in nature $\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$
- Cross product is distributive $\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$
- $\hat{i} \times \hat{j} = \hat{k}; \quad \hat{j} \times \hat{k} = \hat{i}\ ; \quad \hat{k} \times \hat{i} = \hat{j}$
- $\hat{j} \times \hat{i} = -\hat{k}; \quad \hat{k} \times \hat{j} = -\hat{i}\ ; \quad \hat{i} \times \hat{k} = -\hat{j}$
- $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \overline{0}$